

THE QUIVER AT THE BOTTOM OF THE TWISTED NILPOTENT CONE ON \mathbb{P}^1

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Abstract. For the moduli space of Higgs bundles on a Riemann surface of positive genus, critical points of the natural Morse-Bott function lie along the nilpotent cone of the Hitchin fibration and are representations of A-type quivers in a twisted category of holomorphic bundles. The critical points that globally minimize the function are representations of A_1 . For twisted Higgs bundles on the projective line, the quiver describing the bottom of the cone is more complicated. We determine it here. We show that the moduli space is topologically connected whenever the rank and degree are coprime, thereby verifying conjectural lowest Betti numbers coming from high-energy physics.

1. INTRODUCTION

Let X be a Riemann surface and ω_X its canonical line bundle. Recall that a *Higgs bundle* on X is a holomorphic vector bundle E adorned with a holomorphic bundle map $\phi : E \rightarrow E \otimes \omega_X$, usually called a “Higgs field”. From the natural Kähler metric on the moduli space of stable Higgs bundles over a Riemann surface, we can define a Morse-Bott function f that sends each Higgs bundle (E, ϕ) to a scalar multiple of the norm squared of ϕ . The existence of this function enables one to use Morse theory to study the topology of the moduli space — a programme that has been especially successful in low rank ([18, 12, 8, 9] etc.), but which has been difficult to implement in general.

Emerging from this programme is the now well-known fact that the critical set of f is a submanifold of the *nilpotent cone*, which is precisely the locus of Higgs bundles with nilpotent ϕ . When the underlying Riemann surface has genus $g \geq 1$, f attains an absolute minimum of 0, and the submanifold of the nilpotent cone along which $f(E, \phi) = 0$ is precisely that on which ϕ is identically zero.

In other words, the question “What is the *bottom* of the nilpotent cone?” has an easy answer when $g \geq 1$: it is the moduli space of stable bundles, which is embedded into the moduli space of stable Higgs bundles — and in particular, into the nilpotent cone — by $E \mapsto (E, 0)$.

In one sense, the same question for $g = 0$ has an easy answer, since the moduli space of stable Higgs bundles is empty. There are two ways to make this less trivial. One way is to mark the projective line and introduce *parabolic Higgs bundles* adapted to the divisor of the marked points [2]. This has the advantage that the moduli space is not only nonempty but also hyperkähler, like the moduli space of ordinary Higgs bundles on a positive-genus curve. The bottom of the cone is now the moduli space of stable parabolic bundles, assuming it is nonempty.

Another way to coax out a nonempty moduli space of stable Higgs bundles on \mathbb{P}^1 is to consider Higgs fields that take values in an arbitrary ample line bundle $\mathcal{O}(t)$ instead of in ω . The resulting objects are the so-called *twisted Higgs bundles*, studied under various names and mostly in positive genus in [25, 22, 3, 10, 7] for example. At $g = 0$, their moduli space lacks the rich hyperkähler structure associated to ordinary and parabolic Higgs bundle moduli spaces. That being said, there is still a natural Kähler metric, from which we can define a Morse-Bott function

$$f(E, \phi) = \frac{1}{2} \|\phi\|^2.$$

The moduli space retains a *Hitchin fibration*, which is a proper map to an affine base whose generic fibre is a nonsingular abelian variety. As in the ordinary Higgs case, the map sends a Higgs bundle to the characteristic polynomial of its Higgs field. The fibre over zero is an analogue of the global nilpotent variety studied in [11], containing twisted nilpotent Higgs fields in this case. However, this twisted nilpotent cone has no natural relationship to the cotangent bundle of a moduli space of bundles.

The answer to the question about the bottom of the cone is not as immediately clear as in the other settings. First of all, the moduli space of stable bundles is empty when the rank is larger than 1. In other words, there are no stable twisted Higgs bundles of the form $(E, 0)$, and f does not attain 0 as its global minimum. (Note that a similar phenomenon occurs for parabolic Higgs bundles with sufficiently small parabolic weights, as in [8].) This leads to a natural question: what are necessary and sufficient conditions on a twisted Higgs bundle (E, ϕ) on \mathbb{P}^1 for ϕ to minimize f ?

To answer this, we need not interact directly with f . Rather, we know that global minimizers are exactly the points in the moduli space with *Morse index* 0, meaning that there are no further downward directions for the Morse flow. As will be reviewed in §2, a critical point must have a particular form, called a *holomorphic chain*. Their appearances in this context the literature include [13, 4, 1, 26, 32]. The Morse index can be read off directly from the chain.

In representation-theoretic terms, a holomorphic chain is a representation of an A-type quiver Q , but the representations are taken in a category of holomorphic bundles on X with twisted morphisms rather than the category of vector spaces. These objects are a special case of the “quiver bundles” considered in [14, 1, 21, 29]. Here, we consider quivers Q with finite underlying graph A_n for some $n \geq 1$. In Q , the arrows point in the same direction, from left to right, and we number the nodes sequentially, also from left to right. Before we can specify a representation, we fix an auxiliary bundle $F \rightarrow X$ and a labelling of the nodes by pairs of integers r_i, d_i subject to $r_i \geq 0$, $\sum r_i = r$, and $\sum d_i = -d$, where r and d are integers for which $0 < d < r$:

$$\bullet_{r_1, d_1} \longrightarrow \bullet_{r_2, d_2} \longrightarrow \cdots \longrightarrow \bullet_{r_n, d_n}$$

A representation is a $(2n - 1)$ -tuple

$$(U_1, \dots, U_n; \phi_1, \dots, \phi_{n-1})$$

in which U_i is a bundle of rank r_i and degree d_i and ϕ_i is an F -twisted morphism $\phi_i : U_i \rightarrow U_{i+1} \otimes F$.

For the moduli space of ordinary Higgs bundles of rank r and degree $-d$ on a positive-genus curve X , we have $F = \omega_X$ and the global minimizers of f are

representations of the simplest such quiver, A_1 , with the only possible labelling:

$$\bullet_{r,-d}$$

For the correct choice of stability condition, the moduli space of representations associated to this graph is the moduli space of semistable bundles of rank r and degree $-d$. As there are no arrows, the Higgs fields of these bundles are zero.

Finding this quiver for the moduli space of twisted Higgs bundles on \mathbb{P}^1 is equivalent to find its length and labelling. Fix $F = \mathcal{O}(t)$ for some $t > 0$. Let $\mathcal{M}_t(r, -d)$ denote the moduli space of semistable Higgs bundles on \mathbb{P}^1 of rank r and degree $-d$ with $0 < d < r$ and Higgs fields taking values in $\mathcal{O}(t)$; $\text{Nilp}_t(r, -d)$, its nilpotent cone; and $\mathcal{Q}_t(r, -d)$, the moduli space of quiver representations containing the submanifold of $\text{Nilp}_t(r, -d)$ along which f is minimized. We refer to this latter moduli space as a “quiver-bundle variety” to avoid confusion with Nakajima quiver varieties. Our main results are:

Theorem 5.1. When $\gcd(r, d) = 1$, $\mathcal{Q}_t(r, -d)$ is a quiver-bundle variety for the quiver with underlying graph

$$A_{\lceil \log_{t+1}(\frac{r}{d}) \rceil + 1}.$$

If $n = \lceil \log_{t+1}(\frac{r}{d}) \rceil + 1 = 2$, its labels are

$$\bullet_{r-d,0} \longrightarrow \bullet_{d,-d};$$

if $n = \lceil \log_{t+1}(\frac{r}{d}) \rceil + 1 > 2$, then we have

$$\bullet_{r-R,0} \longrightarrow \bullet_{dt(t+1)^{n-3},0} \longrightarrow \cdots \longrightarrow \bullet_{dt,0} \longrightarrow \bullet_{d,-d}$$

where $R = d + dt + \cdots + dt(t+1)^{n-3}$. The submanifold of $\mathcal{Q}_t(r, -d)$ along which f is minimized is the restriction of $\mathcal{Q}_t(r, -d)$ to the following equivalence classes:

$$\{[(U_1, \dots, U_n; \phi_1, \dots, \phi_{n-1})] \mid U_1, \dots, U_{n-1}, U_n \otimes \mathcal{O}(1) \text{ holomorphically trivial,} \\ \phi_1, \dots, \phi_{n-1} \text{ injective}\}.$$

Above, when we ask for ϕ_i to be injective, we mean as a map of global sections.

Theorem 6.1. $\mathcal{M}_t(r, -d)$ is topologically connected whenever $\gcd(r, d) = 1$.

For a large range of r and t values that have been inspected by computer, Theorem 6.1 verifies the conjectural lowest Betti numbers for $\mathcal{M}_t(r, -d)$ coming from Mozgovoy’s twisted version of the ADHM recursion formula [23]. These conjectures can presumably be checked using alternative recent results, namely by extracting the Betti numbers from the Donaldson-Thomas invariants for twisted Higgs bundle moduli spaces computed in [24] or by making appropriate modifications to the arguments for ordinary Higgs bundles over finite fields in [28], so that the closed-form Poincaré series obtained in that paper for ordinary Higgs bundles on Riemann surfaces generalizes to $g = 0$ and twisted Higgs bundles. However, it is satisfying to have a direct, Morse-theoretic proof of the connectedness of the twisted Higgs moduli space on \mathbb{P}^1 in the spirit of Hitchin’s original approach.

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2. MORSE THEORY FOR TWISTED HIGGS BUNDLES

We employ standard notation throughout. In particular, we use $\mathcal{O}(a)$ to denote a representative of the unique isomorphism class of holomorphic line bundles on \mathbb{P}^1 with degree a . Because the degree map is an isomorphism from the multiplicative group $\text{Pic}(\mathbb{P}^1)$ to the additive group of integers, we have $\mathcal{O}(a) \otimes \mathcal{O}(b) \cong \mathcal{O}(a+b)$. The dual of a holomorphic vector bundle E is denoted E^* . With these conventions, $\mathcal{O}(a)^* \cong \mathcal{O}(-a)$. By $\text{End}(E)$, we always mean the bundle $E^* \otimes E$. Its space of global sections, $H^0(\mathbb{P}^1, \text{End}(E)) = H^0(\mathbb{P}^1, E^* \otimes E)$, is precisely the set of all holomorphic bundle maps from E to itself. Normally, we omit the \mathbb{P}^1 in sheaf cohomologies $H^i(\mathbb{P}^1, F)$, as the \mathbb{P}^1 will be understood throughout.

With these conventions, we can formalize what we mean by a twisted Higgs bundle:

Definition 2.1. An $\mathcal{O}(t)$ -twisted Higgs bundle on \mathbb{P}^1 is a pair (E, ϕ) in which E is a holomorphic vector bundle on \mathbb{P}^1 and ϕ is an element of $H^0(\text{End}(E) \otimes \mathcal{O}(t))$. We refer to the integer t as the *twist* of (E, ϕ) .

Throughout the paper, t will be a fixed positive integer, and so we can refer to (E, ϕ) as a “twisted Higgs bundle” without confusion. It is worth noting that when $t = 2$, the Higgs fields are valued in the anticanonical line bundle of \mathbb{P}^1 . These objects are known as *co-Higgs bundles*. They arise in generalized complex geometry and were initially studied in [20, 26, 27].

Definition 2.2. A subbundle $U \subset E$ is ϕ -invariant if $\phi(U) \subseteq U \otimes \mathcal{O}(t)$. The *slope* of U is the rational number $\mu(U) := \deg(U)/\text{rank}(U)$. A twisted Higgs bundle (E, ϕ) is called *semistable* if $\mu(U) \leq \mu(E)$ for all nonzero, proper ϕ -invariant subbundles U of E . If the inequality is strict for all such U , then (E, ϕ) is called *stable*. If (E, ϕ) is not semistable, then it is *unstable*.

This is Hitchin’s slope stability condition from [18] adapted to the twisted Higgs situation. We denote by $\mathcal{M}_t(r, d)$ the moduli space of semistable $\mathcal{O}(t)$ -twisted Higgs bundles (E, ϕ) on \mathbb{P}^1 in which the rank and degree of E are $r > 0$ and d , respectively. It is the set of all semistable pairs (E, ϕ) taken up to the following equivalence: $(E, \phi) \cong (E', \phi')$ if there exists a holomorphic bundle isomorphism $\psi : E \rightarrow E'$ such that $\phi' = \psi^{-1}\phi\psi$.

We will give a rapid summary of the facts surrounding twisted Higgs bundle moduli spaces on \mathbb{P}^1 and the features of Morse theory that apply to them.

- The set $\mathcal{M}_t(r, d)$ is nonempty only when r and t are positive. It carries the structure of a smooth, quasiprojective variety of complex dimension $tr^2 + 1$. Smoothness is guaranteed by the assumption that $\gcd(r, d) = 1$, under which the sets of stable and semistable twisted Higgs bundles of rank r and degree d coincide [25]. The moduli space can be constructed as a GIT quotient [25] or as a Kähler quotient and therefore carries a Kähler metric (by an adaptation of a construction for ordinary, arbitrary-rank Higgs bundles in [30, 31]).

- If $d' \cong d \pmod{r}$, then $\mathcal{M}_t(r, d)$ and $\mathcal{M}_t(r, d')$ are complex-analytically isomorphic (the map is tensoring by a line bundle of appropriate degree), and so it is enough to work with $\mathcal{M}_t(r, -d)$ with d in the range $[0, r)$. Working with nonpositive degree is the convention of our choosing. As we will eventually restrict to $\gcd(r, d) = 1$, we omit $d = 0$ and consider d in the open interval $(0, r)$.
- Consider an $\mathcal{O}(t)$ -twisted Higgs bundle (E, ϕ) . The Birkhoff-Grothendieck Theorem, which classifies holomorphic vector bundles on the projective line up to isomorphism, tells us that $E \cong \bigoplus_{i=1}^r \mathcal{O}(a_i)$ for some unique set of integers a_1, \dots, a_r . This also means that ϕ is globally represented by an $r \times r$ matrix whose (i, j) -th entry is a section ϕ_{ij} of the line bundle $\text{Hom}(\mathcal{O}(a_j), \mathcal{O}(a_i) \otimes \mathcal{O}(t)) = \mathcal{O}(-a_j + a_i + t)$. Note that $\phi^* = \overline{\phi^T}$ is a well-defined $\mathcal{O}(t)$ -valued Higgs field for E^* , and (E, ϕ) is stable if and only if (E^*, ϕ^*) is.
- If a twisted Higgs bundle (E, ϕ) is stable, then the kernel of the map $[-, \phi] : H^0(\text{End}(E)) \rightarrow H^0(\text{End}(E) \otimes \mathcal{O}(t))$ is $\{c\mathbf{1}_E \mid c \in \mathbb{C}\}$. This is a particular case of the general fact that stable objects are *simple*, meaning that endomorphisms of a stable object are generated by the identity. In this case, endomorphisms of (E, ϕ) are endomorphisms of E that commute with ϕ . It follows that, for a stable (E, ϕ) , the map $[-, \phi] : H^0(\text{End}_0(E)) \rightarrow H^0(\text{End}_0(E) \otimes \mathcal{O}(t))$ on trace-free endomorphisms is injective. By duality, we have that the induced map $[-, \phi] : H^1(\text{End}_0(E)) \rightarrow H^1(\text{End}_0(E) \otimes \mathcal{O}(t))$ is surjective.
- There exists a proper map h from $\mathcal{M}_t(r, -d)$ to the affine space $B_{t,r} := \bigoplus_{i=1}^r H^0(\mathcal{O}(it))$, sending a twisted Higgs bundle (E, ϕ) to the r -tuple of characteristic coefficients of ϕ (which are sections of various tensor powers of $\mathcal{O}(t)$). This map is called the *Hitchin map* or *Hitchin fibration*, first introduced for ordinary Higgs bundles in [18, 19]. The fibre $\text{Nilp}_t(r, -d) := h^{-1}(0)$ is referred to as the *nilpotent cone*, as it consists of all (E, ϕ) for which ϕ is nilpotent as a bundle map.
- The function $f(E, \phi) = \frac{1}{2}\|\phi\|^2$, where $\|\cdot\|$ is defined using the Kähler metric, is bounded below and is a perfect Morse-Bott function on $\mathcal{M}_t(r, -d)$. This fact for twisted Higgs bundles adapts without change from [18].
- The flow of f is coincident with $\text{Nilp}_t(r, -d)$ and the critical set of f is a submanifold $\mathcal{C}_f \subset \text{Nilp}_t(r, -d)$. Again, this follows without change from properties of ordinary Higgs bundles presented in [15].
- It follows from [13] that a twisted Higgs bundle (E, ϕ) is a critical point of f if and only if E admits a decomposition $E \cong \bigoplus_{i=1}^n U_i$ for some $n \leq r$, in such a way that $\phi(U_i) \subseteq U_{i+1} \otimes \mathcal{O}(t)$ for each $i = 1, \dots, n-1$ and $\phi(U_n) = 0$. We say that (E, ϕ) has the structure of a *holomorphic chain* of length n . In particular, ϕ is nilpotent of order n . We refer to the bundles U_i as *blocks* of the chain. We say that ϕ acts with *weight* 1 on each block. Note that length $n = 1$ is inadmissible when $r > 1$, as this would correspond to $(E, 0)$, which is always unstable on \mathbb{P}^1 .
- To a holomorphic chain (E, ϕ) , we assign an n -tuple $\mathbf{r} = (r_1, \dots, r_n)$ that consists of the ranks of the blocks.
- The *Morse index* of f at a critical point (E, ϕ) is the number of negative eigenvalues of the Hessian of f at (E, ϕ) . Geometrically, this is the number

of independent downward flow directions of f out of (E, ϕ) . Equivalently, if $N_{(E, \phi)}$ is the normal space to \mathcal{C}_f at (E, ϕ) , then the Morse index of (E, ϕ) is the dimension of the maximal subspace of $N_{(E, \phi)}$ on which $\text{Hess}(f)$ is negative definite. This number is constant on each connected component of \mathcal{C}_f . We will use $\beta(E, \phi)$ to refer to the *complex* Morse index, that is, to the actual Morse index multiplied by $1/2$.

- Fix a holomorphic chain (E, ϕ) with blocks U_1, \dots, U_n , an integer i for which $1 \leq i \leq n$, a nonnegative integer k , and an integer q . Then, put

$$\mathbb{H}_{k,i,q}^p := H^p(U_i^* \otimes U_{i+k} \otimes \mathcal{O}(q))$$

if $k \leq n - i$; otherwise, define it to be the trivial vector space. We refer to elements of $\bigoplus_{i=1}^n \mathbb{H}_{k,i,q}^p$ as (p, q) -endomorphisms of E of *weight* k . (In particular, ϕ itself is a $(0, t)$ -endomorphism of weight 1.) Use $h_{k,i,q}^p$ for the complex dimension of this space.

- The subspace of $N_{(E, \phi)}$ on which the Hessian of f is negative definite is

$$\mathcal{B}(E) \oplus \mathcal{B}(\phi),$$

where

$$\mathcal{B}(E) = \begin{cases} \bigoplus_{i=1}^{n-1} \bigoplus_{k=1}^{n-i} \ker \left(\mathbb{H}_{k,i,0}^1 \xrightarrow{[-, \phi]} \mathbb{H}_{k+1,i,t}^1 \right) & \text{if } n > 1 \\ \{0\} & \text{if } n = 0, 1 \end{cases}$$

and

$$\mathcal{B}(\phi) = \begin{cases} \bigoplus_{i=1}^{n-2} \bigoplus_{k=2}^{n-i} \left(\frac{\mathbb{H}_{k,i,t}^0}{\text{im} \left(\mathbb{H}_{k-1,i,0}^0 \xrightarrow{[-, \phi]} \mathbb{H}_{k,i,t}^0 \right)} \right) & \text{if } n > 2 \\ \{0\} & \text{if } n = 0, 1, 2 \end{cases}$$

cf. [13]. We denote by $\beta(E)$ and $\beta(\phi)$ the complex dimensions of $\mathcal{B}(E)$ and $\mathcal{B}(\phi)$, respectively. In other words, $\beta(E)$ is the subspace of $H^1(\text{End } E)$ consisting of deformations of the complex structure on E that have weight at least 1 (and for which ϕ remains holomorphic), and $\beta(\phi)$ consists of deformations of ϕ of weight at least 2. In this language, the direction of the Morse flow is described by weight spaces within the tangent spaces to $\mathcal{M}_t(r, d)$: the downward flow acts with weight at least 1 on E and with weight at least 2 on the Higgs field; the upward flow acts with weight at most -1 on E and with weight at most 0 on the Higgs field.

- A global minimizer of f is precisely a critical point at which the downward flow terminates. In other words, a critical point (E, ϕ) is a global minimizer of f if and only if $\beta(E, \phi) = 0$, which occurs if and only if $\beta(E) = \beta(\phi) = 0$.

As we noted earlier, the minimum value of f on $\mathcal{M}_t(r, -d)$ — call it f_{\min} — is *positive*, because ϕ is never the zero map. For our purposes, it is easier to classify those (E, ϕ) for which $f(E, \phi) = f_{\min}$ by using the Morse index.

2.1. Calculating the Morse index. In what follows, let δ_n^m be 0 if $m \leq n$ and 1 otherwise. Recall that $[-, \phi] : H^1(\text{End}_0(E)) \rightarrow H^1(\text{End}_0(E) \otimes \mathcal{O}(t))$ is surjective whenever (E, ϕ) is stable. This forces

$$[-, \phi] : \mathbb{H}_{k,i,0}^1 \longrightarrow \mathbb{H}_{k+1,i,t}^1$$

to be surjective, too, whenever $k > 0$. To see this note that, for all $k \geq 0$, elements of $\mathbb{H}_{k+1,i,t}^1 = H^1(U_i^* \otimes U_{i+k+1} \otimes \mathcal{O}(t))$ are trace-free when viewed as twisted endomorphisms of E . Hence, for any $\psi \in \mathbb{H}_{k+1,i,t}^1$, there exists a $\psi_0 \in H^1(\text{End}_0(E))$ for which $[\psi_0, \phi] = \psi$. It is clear that ψ_0 must be an element of $\mathbb{H}_{k,i,0}^1$, as the action of $[-, \phi]$ always increases weights by exactly 1 (i.e. a map from U_i to U_j is sent to a map from U_i to U_{j+1}), and $[-, \phi]$ simultaneously twists by $\mathcal{O}(t)$.

The implication is that

$$\beta(E) = \delta_n^1 \dim_{\mathbb{C}} \bigoplus_{i=1}^{n-1} \bigoplus_{k=1}^{n-i} \ker \left(\mathbb{H}_{k,i,0}^1 \xrightarrow{[-, \phi]} \mathbb{H}_{k+1,i,t}^1 \right) = \delta_n^1 \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} (h_{k,i,0}^1 - h_{k+1,i,t}^1).$$

Each of the differences $h_{k,i,0}^1 - h_{k+1,i,t}^1$ is of course nonnegative, as it is the dimension of a subspace of $\mathbb{H}_{k,i,0}^1$.

For $\beta(E)$, the above expression is all that we will need. For $\beta(\phi)$, we will need a somewhat finer formula for our arguments. Consider the unique Birkhoff-Grothendieck decomposition of E :

$$E \cong \bigoplus_{i=1}^r \mathcal{O}(a_i)$$

for some integers a_i such that $\sum a_i = -d$. After re-indexing the integers so that $a_i \geq a_{i+1}$, we denote this non-increasing sequence as $\text{BG}(E)$. If this sequence is equal to $0, \dots, 0, -1, \dots, -1$, where the number of -1 's in the sequence is d , then we say that E has the *generic type*. (Note that such a sequence is well-defined because $0 < d < r$.)

In particular, if $E \cong \bigoplus_{i=1}^n U_i$, then each summand U_i has its own BG sequence, and the concatenation of $\text{BG}(U_1), \dots, \text{BG}(U_n)$ is a permutation of $\text{BG}(E)$. If $\text{BG}(U_i) = (b_{1,i}, \dots, b_{r_i,i})$, then

$$\mathbb{H}_{k,i,q}^p = H^p(U_i^* \otimes U_{i+k} \otimes \mathcal{O}(q)) \cong \bigoplus_{j=1}^{r_i} H^p(\mathcal{O}(-b_{j,i}) \otimes U_{i+k} \otimes \mathcal{O}(q)).$$

We will use $b_{k,j,i,q}^p$ for $\dim_{\mathbb{C}} H^p(\mathcal{O}(-b_{j,i}) \otimes U_{i+k} \otimes \mathcal{O}(q))$ (and set this to 0 when $i+k > n$). Also, if $u \neq j$ or $v \neq i+k-1$, then the image of the map

$$[-, \phi] : H^p(\mathcal{O}(b_{u,i}) \otimes U_v \otimes \mathcal{O}(q)) \longrightarrow H^p(\mathcal{O}(b_{u,i}) \otimes U_{v+1} \otimes \mathcal{O}(q+t))$$

has empty intersection with $H^p(\mathcal{O}(b_{j,i}) \otimes U_{i+k} \otimes \mathcal{O}(q+t))$.

These considerations imply that

$$\bigoplus_{i=1}^{n-2} \bigoplus_{k=2}^{n-i} \left(\frac{\mathbb{H}_{k,i,t}^0}{\text{im} \left(\mathbb{H}_{k-1,i,0}^0 \xrightarrow{[-, \phi]} \mathbb{H}_{k,i,t}^0 \right)} \right)$$

is equal to

$$\bigoplus_{i=1}^{n-2} \bigoplus_{k=2}^{n-i} \bigoplus_{j=1}^{r_i} \left(\frac{H^0(\mathcal{O}(-b_{j,i}) \otimes U_{i+k} \otimes \mathcal{O}(t))}{\text{im} \left(H^0(\mathcal{O}(-b_{j,i}) \otimes U_{i+k-1}) \xrightarrow{[-, \phi]} H^0(\mathcal{O}(-b_{j,i}) \otimes U_{i+k} \otimes \mathcal{O}(t)) \right)} \right).$$

Also, note that

$$[-, \phi] : H^0(\mathcal{O}(-b_{j,i}) \otimes U_{i+k-1}) \longrightarrow H^0(\mathcal{O}(-b_{j,i}) \otimes U_{i+k} \otimes \mathcal{O}(t))$$

is injective since (E, ϕ) is stable.

It follows that

$$\beta(\phi) = \delta_n^2 \sum_{i=1}^{n-2} \sum_{k=2}^{n-i} \sum_{j=1}^{r_i} (b_{k,j,i,t}^0 - b_{k-1,j,i,0}^0).$$

Again, note that each term $b_{k,j,i,t}^0 - b_{k-1,j,i,0}^0$ is nonnegative, since each is the difference in dimension of two vector spaces, one of which is a subspace of the other. Hence, $\beta(\phi) = 0$ if and only if

$$b_{k,j,i,t}^0 - b_{k-1,j,i,0}^0 = 0$$

for all i, j, k in the appropriate ranges.

3. BUNDLE TYPE AT THE BOTTOM

From now on, (E, ϕ) is a stable holomorphic chain with $\beta(E) = \beta(\phi) = 0$; blocks U_1, \dots, U_n ; and $t > 0$ and $0 < d < r$.

We also let u_m and v_m stand for the number of 0's and -1 's, respectively, in the BG sequence of U_m .

Proposition 3.1. *$E = \bigoplus_{i=0}^n U_i$ has the generic type, $U_1 \cong \mathcal{O}^{\oplus r_1}$, and $U_n \cong \mathcal{O}(-1)^{\oplus r_n}$, where $r_n \leq d$.*

Remark. The claim implies that $r_m = u_m + v_m$ for each block and in particular $u_1 = r_1$, $v_1 = 0$, $u_n = 0$, $v_n = r_n$.

Proof. Since a chain length of 1 is prohibited by stability when $r > 1$, we can assume that $n \geq 2$. For (E, ϕ) to be a global minimizer, we must have $\beta(E) = 0$, which is equivalent to

$$\sum_{i=0}^{n-1} \sum_{k=1}^{n-i} h^1(U_i^* \otimes U_{i+k}) = \sum_{i=0}^{n-1} \sum_{k=1}^{n-i} h^1(U_i^* \otimes U_{i+k+1} \otimes \mathcal{O}(t)).$$

By Serre duality, the left side of this is

$$\mathcal{L} := \sum_{i=0}^{n-1} \sum_{k=1}^{n-i} h^0(U_i \otimes U_{i+k}^* \otimes \mathcal{O}(-2))$$

and the right-hand side is

$$\mathcal{R} := \sum_{i=0}^{n-1} \sum_{k=1}^{n-i} h^0(U_i \otimes U_{i+k+1}^* \otimes \mathcal{O}(-t-2)).$$

For purposes of comparison, we re-index the latter sum:

$$\mathcal{R} := \sum_{i=0}^{n-2} \sum_{k=2}^{n-i} h^0(U_i \otimes U_{i+k}^* \otimes \mathcal{O}(-t-2)).$$

Note that \mathcal{R} is nonzero precisely when there are an i and a k such that U_i contains a sub-line bundle isomorphic to $\mathcal{O}(a)$ and U_{i+k} contains a sub-line bundle isomorphic to $\mathcal{O}(b)$ with $a \geq b + t + 2$. Since t is positive, this means that $a > b + 2$. In turn, this implies that

$$h^0(U_i \otimes U_{i+k}^* \otimes \mathcal{O}(-2)) > h^0(U_i \otimes U_{i+k}^* \otimes \mathcal{O}(-t-2)) > 0,$$

and so \mathcal{L} will not only be positive, but also *strictly larger* than \mathcal{R} . In other words, $\mathcal{L} = \mathcal{R}$ if and only if $\mathcal{L} = \mathcal{R} = 0$. Moreover, since \mathcal{L} is larger than \mathcal{R} whenever \mathcal{R} is nonzero, this means that it is sufficient for us to determine the bundles E for which $\mathcal{L} = 0$.

It follows that a necessary condition for having $\mathcal{L} = 0$ is that for each e in the BG decomposition of U_i and each f in the decomposition of U_j with $j \geq i + 1$, we must have $e \leq f + 1$.

The block U_n is annihilated by ϕ and hence is ϕ -invariant, and so every sub-line bundle of U_n must have slope less than $-d/r$. The slope of a sub-line bundle is just its degree, and so we must have $f \leq -1$ for each f in the BG decomposition of U_n . If some f in the decomposition of U_n is less than or equal to -2 , then this every number in the BG decomposition of E is less than or equal to $-2 + 1 = -1$ (by the condition for having $\mathcal{L} = 0$), which contradicts the fact that $0 < d < r$.

Hence, every number in the BG decomposition of U_n is -1 , which proves the part of the proposition that says that $U_n \cong \mathcal{O}(-1)^{\oplus r_n}$.

Comparing U_1 and U_n , the $\mathcal{L} = 0$ condition says that every number e in the BG decomposition of U_1 must satisfy $e \leq -1 + 1 = 0$. In the dual chain (E^*, ϕ^*) , which has slope d/r , U_1^* is the block that is annihilated by ϕ^* , and so we must have $-e < d/r$ for all e in the decomposition of U_1 . If $e < 0$, then $-e < d/r$ forces d/r to be larger than 1, which is a contradiction. Hence, every number in the BG decomposition of U_1 is 0, which proves the part of the proposition that says that $U_1 \cong \mathcal{O}^{\oplus r_1}$ for some $r_1 < r$.

If $n = 2$, then we are done, as this implies that $r_1 + r_2 = r$ and the BG decomposition of E is $0, \dots, 0, -1, \dots, -1$ (with r_1 -many 0's and r_2 -many -1 's). If $n > 2$, then let U_j be any block with $1 < j < n$. First of all, if e is a number in the BG decomposition of U_j , then we must have $e \leq -1 + 1 = 0$ (by invoking the $\mathcal{L} = 0$ condition and comparing to U_n). If e were at most -2 , then every number in the decomposition of U_1 would necessarily be bounded above by -1 , which contradicts the fact that the decomposition of U_1 contains only zeroes. Hence, we must have that e is either -1 or 0.

Hence, every number in the BG decomposition of E is either a -1 or a 0, and so there must be exactly d -many -1 's. In other words, E must be of generic type. This also forces the number r_n to be less than or equal to d . \square

We sharpen the description of the blocks a little more now.

Lemma 3.1. *Let $n \geq 4$. Then $v_1, \dots, v_{n-3} = 0$, and v_{n-2} and v_{n-1} cannot be simultaneously nonzero. Moreover, when $v_{n-2} \neq 0$, we must have $t = 1$.*

Proof. First of all, by Proposition 3.1, we know that $v_1 = 0$ always. We begin by assuming $n > 4$ and choose any U_m with m in $2 \leq m \leq n - 3$. First, because $\beta(\phi) = 0$, we must have

$$b_{j,1,n-1,t}^0 - b_{j,1,n-2,0}^0 = 0$$

for each j in $1 \leq j \leq r_1$, and so

$$h^0(\mathcal{O}(-b_{j,m}) \otimes U_n \otimes \mathcal{O}(t)) - h^0(\mathcal{O}(-b_{j,m}) \otimes U_{n-1}) = 0.$$

Since $U_1 \cong \mathcal{O}^{\oplus r_1}$, $U_{n-1} = \mathcal{O}^{\oplus u_{n-1}} \oplus \mathcal{O}(-1)^{\oplus v_{n-1}}$, and $U_n \cong \mathcal{O}(-1)^{\oplus r_n}$, the equation becomes

$$r_n(-0 + (-1) + t + 1) - u_{n-1}(-0 + 0 + 1) - v_{n-1}(-0 + (-1) + 1) = 0,$$

from which we get $u_{n-1} = r_n t$. It is also necessary that

$$b_{j,m,n-m,t}^0 - b_{j,m,n-m-1,0}^0 = 0$$

for each j in $1 \leq j \leq r_m$. Assume that the BG sequence of U_m contains a -1 and choose j so that $b_{j,m} = -1$. The condition

$$b_{j,m,n-m,t}^0 - b_{j,m,n-m-1,0}^0 = 0$$

becomes

$$r_n(-(-1) + (-1) + t + 1) - u_{n-1}(-(-1) + 0 + 1) - v_{n-1}(-(-1) + (-1) + 1) = 0.$$

Combining this with $u_{n-1} = r_n t$, we obtain $v_{n-1} = r_n(1 - t)$. This forces $v_{n-1} = 0$ and $t = 1$.

Two additional conditions for $\beta(\phi) = 0$ are

$$b_{j,1,n-2,t}^0 - b_{j,1,n-3,0}^0 = 0$$

and

$$b_{j,m,n-m-1,t}^0 - b_{j,m,n-m-2,0}^0 = 0.$$

Now that $t = 1$, $v_{n-1} = 0$, and $u_{n-1} = r_n$, the first of these conditions becomes $u_{n-2} = 2r_n$ and the second becomes $3u_{n-1} = 2u_{n-2} + v_{n-2}$. Combining them, we get $v_{n-2} = -r_n$, which is a contradiction since $r_n > 0$. Hence, $\text{BG}(U_m)$ cannot contain a -1 if $2 \leq m \leq n-1$.

In the case of $n = 4$, we only have $v_{n-1} = v_3$ and $v_{n-2} = v_2$ to be concerned with. Assume $v_2 \neq 0$. The conditions $b_{j,1,n-1,t}^0 - b_{j,1,n-2,0}^0 = 0$ for any j in $1 \leq j \leq r_1$ and $b_{\ell,2,n-2,t}^0 - b_{\ell,2,n-3,0}^0 = 0$ for any ℓ for which $b_{\ell,j} = -1$, we get $v_3 = r_4(1 - t)$ which implies that $v_3 = 0$ and $t = 1$. □

An immediate consequence of Lemma 3.1 is that either $d = r_n + v_{n-1}$ or $d = r_n + v_{n-2}$ when $r \geq 4$, $d = r_3 + v_2$ when $r = 3$, and $d = r_n$ when $r = 2$.

Ruling out the positivity of v_{n-1} and v_{n-2} requires more work and uses stability.

4. HIGGS FIELDS AT THE BOTTOM

We are now prepared to prove one of the main theorems. Note that there exists a unique nonnegative integer N determined by r : if $d < r \leq d + dt$, then $N = 0$; otherwise, N is the positive integer for which

$$d + dt + \cdots + dt(t+1)^{N-1} < r \leq d + dt + \cdots + dt(t+1)^N.$$

The proof of Theorem 4.1 uses this number and is divided into cases, but the idea is the same in each case: to locate a destabilizing ϕ -invariant subbundle whenever ϕ does not take a particular form.

Theorem 4.1. *If $d < r \leq d + dt$, then*

$$\mathbf{r} = (r - d, d).$$

If $r > d + dt$, then

$$\mathbf{r} = (r - R, dt(t+1)^{n-3}, \dots, dt(t+1), dt, d),$$

where

$$R = d + dt + dt(t+1) + \cdots + dt(t+1)^{n-3}$$

and

$$r - R \leq dt(t+1)^{n-2}.$$

Proof. Recall by Lemma 3.1 that v_{n-1} and v_{n-2} are the only numbers v_m that can be nonzero. We start by assuming that in any case where v_{n-2} is well-defined, i.e. for $n \geq 3$, we have $v_{n-2} = 0$. (When $n = 3$, $v_{n-2} = v_1 = 0$ by Proposition 3.1 directly, but for $n \geq 4$ this is not yet obvious.) This means that $d = r_n + v_{n-1}$.

Now assume that (E, ϕ) has $n \geq N + 3 \geq 4$ blocks. It is a consequence of $\beta(\phi) = 0$ that

$$u_{n-1} = r_n t, \quad u_{n-2} = r_n t(t+1) + v_{n-1} t, \dots, u_{n-(N+1)} = r_n t(t+1)^N + v_{n-1} t(t+1)^{N-1}.$$

The space of global sections of $U_{n-1} \otimes \mathcal{O}(t)$ can, after a choice of isomorphism, be identified with $\mathbb{C}^{r_n t(t+1)} \oplus \mathbb{C}^{v_{n-1} t}$. The space of global sections of U_{n-2} can be identified with the same vector space. Note that ϕ_{n-2} , considered as a map of global sections, must be injective; otherwise, its kernel is the space of sections of an invariant, destabilizing trivial subbundle. Hence, ϕ_{n-2} , as a map of global sections, is an isomorphism of vector spaces. Now, consider the subbundle U'_{n-1} of U_{n-1} that is isomorphic to $\mathcal{O}^{\oplus r_n t}$, and take $\phi^{-1}(H^0(U'_{n-1} \otimes \mathcal{O}(t)))$, which is a vector subspace of $H^0(U_{n-2})$ of dimension $r_n t(t+1)$. Since U_{n-2} contains no positive-degree subbundles, $\phi^{-1}(H^0(U'_{n-1} \otimes \mathcal{O}(t)))$ must be the space of sections of a subbundle isomorphic to $\mathcal{O}^{\oplus r_n t(t+1)}$. Let U'_{n-2} be this subbundle. Continue in this way by defining $U'_{n-m} \cong \mathcal{O}^{\oplus r_n t(t+1)^{m-1}}$ to be the subbundle of U_{n-m} whose space of global sections is the preimage of $H^0(U'_{n-m+1} \otimes \mathcal{O}(t))$ under ϕ_{n-m} , which again is injective as a map of global sections. In this way, we get a proper ϕ -invariant subbundle U of E :

$$U = U_n \oplus U'_{n-1} \oplus \dots \oplus U'_{n-(N+1)},$$

which is isomorphic to

$$\mathcal{O}(-1)^{\oplus r_n} \oplus \mathcal{O}^{\oplus r_n t} \oplus \dots \oplus \mathcal{O}^{\oplus r_n t(t+1)^N}.$$

Its slope is

$$\mu(U) = \frac{-r_n}{r_n + r_n t + \dots + r_n t(t+1)^N} = \frac{-d}{d + dt + \dots + dt(t+1)^N} \geq \frac{-d}{r},$$

and so U is destabilizing. This bundle is always proper and destabilizing when $n \geq N + 3$, even if $v_{n-1} = 0$, and so it follows that n can be most $N + 2$. If $n = N + 1$, then $u_2 = u_{n-(N-1)} = r_n t(t+1)^{N-2} + v_{n-1} t(t+1)^{N-3}$ and by the injectivity of ϕ_1 , $r_n \leq r_n t(t+1)^{N-1} + v_{n-1} t(t+1)^{N-2}$, and so we have a contradiction with $d + dt + \dots + dt(t+1)^{N-1} < r$.

Hence, $n = N + 2$. Again, ϕ_1 is necessarily injective and so

$$r_1 \leq r_n t(t+1)^N + v_{n-1} t(t+1)^{N-1}.$$

Recalling that $d + dt + \dots + dt(t+1)^{N-1} < r$, we must also have $r_1 > v_{n-1} t(t+1)^{N-1}$. However, because $r \leq d + dt + \dots + dt(t+1)^N$, we have a contradiction if $r > d + dt + \dots + dt(t+1)^{N-1} + r_n t(t+1)^N$, which is resolved only if $v_{n-1} = 0$. So now restrict to the range $r \leq d + dt + \dots + d(t+1)^{N-1} + r_n t(t+1)^N$. We can write r_1 as $K + v_{n-1} t(t+1)^{N-1}$ where $1 \leq K \leq r_n t(t+1)^N$. Define in the same way as above a subbundle U , but take $U'_1 \cong \mathcal{O}^{\oplus K}$ to be a subbundle of U_1 whose global sections lie in the preimage of $H^0(U'_2 \otimes \mathcal{O}(t)) \cong \mathbb{C}^{r_n t(t+1)^N}$ under ϕ_1 . Note that $K/r_n > K/d$ and that

$$\mu(E) = \frac{-d}{r} = \frac{-d}{d + dt + \dots + dt(t+1)^{N-1} + K} = \frac{-1}{1 + t + \dots + t(t+1)^{N-1} + \frac{K}{d}}.$$

Comparatively,

$$\mu(U) = \frac{-r_n}{r_n + r_n t + \cdots + r_n t(t+1)^{N-1} + K} = \frac{-1}{1 + t + \cdots + t(t+1)^{N-1} + \frac{K}{r_n}}.$$

It follows immediately that U is destabilizing, unless $v_{n-1} = 0$, in which case U is no longer proper.

Now we have $v_{n-1} = 0$ and it now follows that $r_m = u_m$ for $1 \leq m \leq n-1$ and $r_n = d$, and so \mathbf{r} is as described in the statement of the theorem.

Finally, we want to eliminate completely the case of $v_{n-2} > 0$ when $n \geq 4$. To do this, we assume $n \geq 4$ and that v_{n-2} is nonzero and seek contradictions. First, we must have $v_{n-1} = 0$ and $t = 1$ by Lemma 3.1, and so r lies in either the range

$$d < r \leq 2d,$$

for which $N = 0$, or

$$d + 2^0 d + \cdots + 2^{N-1} d < r \leq d + 2^0 d + \cdots + 2^N d.$$

Assume that $n \geq N + 3$. The $\beta(\phi) = 0$ conditions manifest themselves as

$$u_{n-1} = r_n, \quad u_{n-2} = 2r_n, \quad u_{n-3} = 4r_n + v_{n-2}, \dots, \quad u_{n-(N+1)} = 2^N r_n + 2^{N-2} v_{n-2}.$$

We define a proper ϕ -invariant subbundle

$$U = U_n \oplus U_{n-1} \oplus U'_{n-2} \oplus \cdots \oplus U'_{n-(N+1)},$$

where U'_{n-2} is the subbundle of U_{n-2} isomorphic to $\mathcal{O}^{\oplus 2r_n}$, and further U'_{n-m} are defined as above to be the subbundle of U_{n-m} whose space of global sections is the preimage of those of $U'_{n-m+1} \otimes \mathcal{O}(t)$ under ϕ_{n-m} . As above, $U'_{n-m} \cong \mathcal{O}^{\oplus 2^{m-1} r_n}$. The slope is

$$\mu(U) = \frac{-r_n}{r_n + r_n + 2r_n + \cdots + 2^N r_n} = \frac{-d}{d + 2^0 d + \cdots + 2^N d} \geq \frac{-d}{r}$$

and so this bundle is destabilizing. This bundle is always proper and destabilizing when $n \geq N + 3$, even if $v_{n-2} = 0$, and so we must have $n \leq N + 2$. As in the preceding arguments, it is easy to eliminate values of n smaller than $N + 2$. Since $n \geq 4$ by assumption, we must have N at least 2.

It follows from the injectivity of ϕ_1 that $r_1 \leq 2^N r_n + 2^{N-2} v_{n-2}$. This means that we cannot have

$$r > d + 2^0 d + \cdots + 2^{N-2} d + 2^{N-1} r_n + 2^N r_n,$$

and so for such r , we have a contradiction. So now we restrict to

$$d + 2^0 d + \cdots + 2^{N-1} d < r \leq d + 2^0 d + \cdots + 2^{N-2} d + 2^{N-1} r_n + 2^N r_n.$$

This provides the further restriction $2^{N-1} d < (2^{N-1} + 2^N) r_n$, which is equivalent to $d < 3r_n$. We can write $r_1 = K + 2^{N-2} v_{n-2}$ where $1 \leq K \leq 2^N r_n$. Define in the same way as above a subbundle U , but take $U'_1 \cong \mathcal{O}^{\oplus K}$ to be a subbundle of U_1 whose global sections lie in the preimage of $H^0(U'_2 \otimes \mathcal{O}(1)) \cong \mathbb{C}^{2^N r_n}$ under ϕ_1 . Note that $K/r_n > (K - 2^{N-1} v_{n-2})/d$ and that

$$\begin{aligned} \mu(E) &= \frac{-d}{d + 2^0 d + \cdots + 2^{N-1} d + (K - 2^{N-1} v_{n-2})} \\ &= \frac{-1}{1 + t + \cdots + t(t+1)^{N-1} + \frac{K}{d}}. \end{aligned}$$

On the other hand,

$$\mu(U) = \frac{-r_n}{r_n + r_n t + \cdots + r_n t(t+1)^{N-1} + K} = \frac{-1}{1 + t + \cdots + t(t+1)^{N-1} + \frac{K}{r_n}}.$$

It follows immediately that U is destabilizing, unless $v_{n-2} = 0$, in which case U is no longer proper.

Having eliminated $v_{n-2} > 0$ in every possible case, we default to the preceding arguments and so ϕ has the claimed shape. \square

Remark. The $d = 1$ case of this result appears in [26]. The theorem for that particular case is markedly easier to establish than for general d . One reason is that for $d = 1$ it is possible to use an inductive argument based on the rank. If we delete a line bundle from U_1 and then restrict the Higgs field to the resulting rank $r - 1$ bundle, the new Higgs bundle is a stable minimizer. Its stability is ensured because the distribution of -1 's amongst the blocks is already known: there is only one -1 and so it must be in U_n (which is then just $\mathcal{O}(-1)$ itself) and so every proper invariant subbundle will have slope $-1/r' < -1/(r - 1)$. It is also true for general d coprime to r that this restriction procedure produces successive stable minimizers, but this is only known *a posteriori*, after proving Theorem 4.1.

With Theorem 4.1 come the following immediate corollaries:

Corollary 4.1. *If (E, ϕ) is a stable global minimizer of f , then U_1, \dots, U_{n-1} and $U_n \otimes \mathcal{O}(1)$ are holomorphically trivial.*

Corollary 4.2. *Any two stable global minimizers of f for the same r, d , and t have equal lengths and equal rank vectors and their corresponding blocks are isomorphic as holomorphic bundles.*

Conversely, it is easy to check that any stable (E, ϕ) with structure specified by Theorem 4.1 and Corollary 4.1 satisfies $\beta(E) = \beta(\phi) = 0$. Such a Higgs bundle is stable precisely when the maps $\phi_1, \dots, \phi_{n-1}$ are injective as maps of global sections. That this is necessary is a consequence of the proof of Theorem 4.1. (There, we saw that ϕ_i restricted to any holomorphically-trivial subbundle of U_i must induce an injective map of global sections. We now know that U_1, \dots, U_{n-1} are holomorphically trivial themselves.) One can show this is sufficient as well, by using the injectivity in combination with an appropriate automorphism of U_n (applied to ϕ_{n-1}) to show that every invariant subbundle has maximally-negative degree $-d$.

Although stability has been assumed all along, we include that hypothesis explicitly in Corollary 4.2 to emphasize that the statement is not necessarily true when $\gcd(r, d) \neq 1$, as there may be minimizers that are semistable but not stable and which do not take the shape prescribed by Theorem 4.1. Case in point, consider $r = 4, d = 2$, and $t = 1$. Any critical point (E, ϕ) with $U_1 \cong \mathcal{O}$, $U_2 \cong \mathcal{O} \oplus \mathcal{O}(-1)$, and $U_3 \cong \mathcal{O}(-1)$ is at best semistable, as there is always an invariant subbundle of degree $-1/2$. (This Higgs bundle has $v_{n-1} \neq 0$ but does not violate the proof of Theorem 4.1 because it is not strictly stable.) An example Higgs field for this bundle that attains semistability is the one that maps U_1 identically onto $\mathcal{O}(-1) \otimes \mathcal{O}(1) \subset U_2 \mathcal{O}(1)$, $\mathcal{O} \subset U_2$ identically onto $\mathcal{O}(-1) \otimes \mathcal{O}(1) \subset U_3 \otimes \mathcal{O}(1)$, and $\mathcal{O}(-1) \subset U_2$ to $\mathcal{O}(-1) \otimes \mathcal{O}(1) \subset U_3 \otimes \mathcal{O}(1)$ via a choice of nonzero section

$p \in H^0(\mathcal{O}(1))$. One can check that this example satisfies $\beta(E) = 0$ and $\beta(\phi) = 0$ by appealing directly to their definitions (as opposed to the formulas derived in Section (2.1), which depend on strict stability).

Returning to strictly stable minimizers, observe that we can express the length n in terms of r , d , and t . If $N \geq 1$, we can take the sum of the geometric series with common ratio $t + 1$ and write

$$(t + 1)^N < \frac{r}{d} \leq (t + 1)^{N+1}.$$

Using the fact that $n = N + 2$, we get

$$n < \log_{t+1} \left(\frac{r}{d} \right) + 2 \leq n + 1,$$

from which we obtain a closed-form formula for n as a function of r, d, t :

$$n(r, d, t) = \left\lceil \log_{t+1} \left(\frac{r}{d} \right) \right\rceil + 1.$$

Notice that this formula produces all the relevant values of n , for when $d < r \leq d + dt$ this formula returns 2.

Taken all together, we arrive at the following theorem:

Theorem 4.2. *Let t, d, r be positive integers for which $0 < d < r$. If*

$$\left\lceil \log_{t+1} \left(\frac{r}{d} \right) \right\rceil + 1 = 2,$$

then a stable Higgs bundle (E, ϕ) is a global minimizer of f in $\mathcal{M}_t(r, -d)$ if and only if

- $E \cong U_1 \oplus U_2$
- $\text{rk}(U_1) = r - d$ and $\text{rk}(U_2) = d$
- U_1 and $U_2 \otimes \mathcal{O}(1)$ are holomorphically trivial
- $\phi(U_1) \subseteq U_2 \otimes \mathcal{O}(t)$ and ϕ is injective as a map of global sections.

If

$$\left\lceil \log_{t+1} \left(\frac{r}{d} \right) \right\rceil + 1 = n > 2,$$

then a stable Higgs bundle (E, ϕ) is a global minimizer of f in $\mathcal{M}_t(r, -d)$ if and only if

- $E \cong \bigoplus_{i=1}^n U_i$
- $\text{rk}(U_n) = d$, $\text{rk}(U_i) = dt(t + 1)^{n-i-1}$ for $2 \leq i \leq n - 1$, and $\text{rk}(U_1) = r - (d + dt + dt(t + 1) + \cdots + dt(t + 1)^{n-3})$
- U_1, \dots, U_{n-1} and $U_n \otimes \mathcal{O}(1)$ are holomorphically trivial
- $\phi(U_i) \subseteq U_{i+1} \otimes \mathcal{O}(t)$ for $1 \leq i \leq n - 1$ and $\phi(U_n) = 0$
- $\phi_1, \dots, \phi_{n-1}$ are injective as maps of global sections. If $\gcd(r, d) = 1$, then all minimizers are of the form described above.

5. THE QUIVER

Theorem 4.2 encodes the quiver $\mathcal{Q}_t(r, -d)$ whose moduli space of representations contains all the stable minimizers of f in $\text{Nilp}_t(r, -d) \subset \mathcal{M}_t(r, -d)$, and also tells us the locus in $\mathcal{Q}_t(r, -d)$ along which $f(E, \phi) = f_{\min}$. The following is a rephrasing of Theorem 4.2:

Theorem 5.1. *When $\gcd(r, d) = 1$, $\mathcal{Q}_t(r, -d)$ is a quiver-bundle variety for the quiver with underlying graph*

$$A_{\lceil \log_{t+1}(\frac{r}{d}) \rceil + 1}.$$

If $n = \lceil \log_{t+1}(\frac{r}{d}) \rceil + 1 = 2$, its labels are

$$\bullet_{r-d,0} \longrightarrow \bullet_{d,-d};$$

if $n = \lceil \log_{t+1}(\frac{r}{d}) \rceil + 1 > 2$, then we have

$$\bullet_{r-R,0} \longrightarrow \bullet_{dt(t+1)^{n-3},0} \longrightarrow \cdots \longrightarrow \bullet_{dt,0} \longrightarrow \bullet_{d,-d}$$

where $R = d + dt + \cdots + dt(t+1)^{n-3}$. The submanifold of $\mathcal{Q}_t(r, -d)$ along which f is minimized is the restriction of $\mathcal{Q}_t(r, -d)$ to the following equivalence classes:

$$\{[(U_1, \dots, U_n; \phi_1, \dots, \phi_{n-1})] \mid U_1, \dots, U_{n-1}, U_n \otimes \mathcal{O}(1) \text{ holomorphically trivial,} \\ \phi_1, \dots, \phi_{n-1} \text{ injective}\}.$$

When $\gcd(r, d) \neq 1$ (for example, when $r = d + dt + \cdots + dt(t+1)^N$ and $d > 1$), there may be semistable but not stable minimizers that do not correspond to the quiver above. In the previous section, we saw that $\mathcal{M}_1(4, 2)$ has a minimizer corresponding to the quiver

$$\bullet_{1,0} \longrightarrow \bullet_{2,-1} \longrightarrow \bullet_{1,-1}$$

The key difference is that this quiver has an interior node of nonzero degree.

6. COMPONENTS AND THE ADHM RECURSION FORMULA

By Hitchin's Morse-theoretic localization procedure [18], the Poincaré polynomial for the ordinary rational cohomology of the moduli space is

$$\mathcal{P}(\mathcal{M}_t(r, -d); y) = \sum_i y^{2\beta(\mathcal{N}_i)} \mathcal{P}_y(\mathcal{N}_i),$$

where \mathcal{N}_i is the i -th connected component of the critical set of f , $\beta(\mathcal{N}_i)$ is the complex Morse index of any Higgs bundle in \mathcal{N}_i , and $\mathcal{P}_y(\mathcal{N}_i)$ is the Poincaré polynomial of \mathcal{N}_i in the variable y . It is worth noting that, because of the properness of the Hitchin fibration and hence the compactness of $\text{Nilp}_t(r, -d)$, there are only finitely-many critical components.

The constant term of $\mathcal{P}(\mathcal{M}_t(r, -d); y)$ is the number of connected components of $\mathcal{M}_t(r, -d)$. It is clear that this number of connected components of the set of global minimizers of the Morse-Bott function. Having classified the solutions of $f(E, \phi) = f_{\min}$ when $\gcd(r, d) = 1$, we can count these components.

Theorem 6.1. *Assume $\gcd(r, d) = 1$. As a complex variety, the space of global minimizers of f is isomorphic to the Grassmannian*

$$\text{Gr}(r - d, dt)$$

when $n = \lceil \log_{t+1}(\frac{r}{d}) \rceil = 2$, and is isomorphic to

$$\text{Gr}(r - R, dt(t+1)^{n-2}),$$

where $R = d + dt + dt(t+1) + \cdots + dt(t+1)^{n-3}$, when $n = \lceil \log_{t+1}(\frac{r}{d}) \rceil + 1 > 2$. As such, $\mathcal{M}_t(r, -d)$ is topologically connected.

Proof. By Theorem 4.2, the blocks U_1, \dots, U_n of the Higgs bundles that minimize f have fixed holomorphic type, and so the moduli are concentrated in the Higgs fields. The condition on the Higgs fields is that they are injective as maps of sections. For $\phi_i : U_i \rightarrow U_{i+1} \otimes \mathcal{O}(t)$ with $i > 1$, this requires that ϕ_i is an isomorphism of spaces of global sections, as $H^0(U_i)$ and $H^0(U_{i+1} \otimes \mathcal{O}(t))$ always have the same dimension (also by Theorem 4.2). Noting that ϕ_i as a bundle map is determined by its induced map on global sections of $U_i \cong \mathcal{O}^{\oplus r_i}$, we have a one-to-one correspondence between admissible maps ϕ_i and elements of $\mathrm{GL}(r_i, \mathbb{C})$ for each $i > 1$. Each of these maps of global sections is acted on the right by $\mathrm{Aut}(U_i) \cong \mathrm{GL}(r_i, \mathbb{C})$. The quotient is just $\{\mathrm{I}\} \in \mathrm{GL}(r_i, \mathbb{C})$. What remains is ϕ_1 . This is an injective map from $H^0(U_1)$ to $H^0(U_2 \otimes \mathcal{O}(t))$, but now $h^0(U_1) \leq h^0(U_2 \otimes \mathcal{O}(t))$ by Theorem 4.2. In other words, ϕ_1 is in correspondence with embeddings of an r_1 -plane into $H^0(U_2 \otimes \mathcal{O}(t))$. Quotienting by the right multiplication action of $\mathrm{Aut}(U_1) \cong \mathrm{GL}(r_1, \mathbb{C})$ gives us a Grassmannian. This is one of the two Grassmannians in the statement of the theorem, depending on whether $n = 2$ (in which case U_2 is $\mathcal{O}(-1)^{\oplus d}$) or $n > 2$ (in which case U_2 is trivial). \square

In particular, the submanifold of stable minimizers has dimension

$$r_1(dt(t+1)^{n-2} - r_1),$$

where $r_1 = r - (d + dt + \dots + dt(t+1)^{n-3})$ when $n > 2$, $r_1 = r - d$ when $n = 2$, and $1 \leq r_1 \leq dt(t+1)^{n-2}$. There is a unique stable global minimizer up to isomorphism when

$$r = d + dt + \dots + dt(t+1)^{n-2}.$$

In particular, when $d = 1$, there is a unique global minimizer up to isomorphism for each of these ranks. When $d \neq 0$, these ranks are not coprime to d , and there will in general be semistable minimizers in addition to the unique stable one.

Finally, we remark that our calculation of the lowest nonzero Betti number of $\mathcal{M}_t(r, -d)$, which is always 1 whenever $\mathrm{gcd}(r, d) = 1$, verifies the conjectural lowest Betti number coming from the twisted ADHM recursion formula for a large range of ranks and twists that can be checked by computer. The twisted ADHM recursion formula was posed and studied by Mozgovoy [23] as a generalization of the Chuang-Diaconescu-Pan ADHM recursion formula coming from physics [6] (see also [5]). Regarding its solutions, we should add that the ADHM Betti numbers depend on two parameters which can be identified with r and t . There is no dependence on d , which is consistent with the fact that the Betti numbers of ordinary Higgs bundle moduli spaces are independent of the degree, as proved in [16], at least when $\mathrm{gcd}(r, d) = 1$. The proof in [16] relies on the homeomorphism to a character variety induced by nonabelian Hodge theory, which does not apply for non-parabolic twisted Higgs bundles on \mathbb{P}^1 , and so this invariance is merely conjectural for such twisted Higgs bundles.

We should note, however, that we already have an example of an instance where the moduli space is not connected when r and d are not coprime. As above demonstrated above, $\mathcal{M}_2(4, 2)$ not only has minimizers of the form prescribed by Theorem 4.2, but also semistable minimizers as elicited in the example following Corollary 4.2. It is known that the degree of a block U_i is constant on a connected component of the critical set of f (Lemma 9.2 in [17], attributed to C. Simpson), and so these two types of minimizers cannot belong to the same component, as the degree of U_2 and U_3 in one do not respectively match those of U_2 and U_3 in the other.

This disconnects the moduli space, and gives an explicit example of how the degree invariance of the Poincaré polynomial fails when $\gcd(r, d)$ is left unrestricted.

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